

# On some criterions of finiteness conditions in commutative Moufang loops

A. D. Babiy, N. I. Sandu

23.04.2008

## Abstract

The various finiteness conditions in commutative Moufang loops are characterized using the notions of centralizer of subloops and centralizer of subgroups of its multiplication group.

**Mathematics subject classification:** 20N05.

**Keywords and phrases:** commutative Moufang loop, multiplication group, centralizer, finite generated, special rank, minimum condition for subloops, maximum condition for subloops.

The (*special*) *rank* of loop  $L$  is called the least positive number  $rL$  with the following feature: any finitely generated subloop of loop  $L$  can be generated by  $rL$  elements; if there are not such numbers, then we suppose that  $rL = \infty$ . Let  $\Omega$  denote one of following classes of loops: the class of finite loops; the class of finitely generated loops; the class of loops of finite rank; the class of loops with maximum condition for its subloops; the class of loops with minimum condition for its subloops.

It is known that in some classes of groups, loops various finiteness conditions of its centre are transferred on these groups, loops. For example, in [1] it is proved that if the centre of finitely generated nilpotent group is finite, then the group itself is finite. Further, for a commutative Moufang  $ZA$ -loop  $L$  with multiplication group  $\mathfrak{M}$  from paper [2 - 4] it follows the equity of statements: 1)  $L$  belongs to class  $\Omega$ ; 2) the centre of  $L$  belongs to class  $\Omega$ ; 3)  $\mathfrak{M}$  belongs to class  $\Omega$ ; 4) the centre of  $\mathfrak{M}$  belongs to class  $\Omega$ .

There exists a commutative Moufang loop (CML) with trivial centre [5]. Then for the described CML with finiteness conditions it is reasonable to use the notion of centralizer, more general than the notion of centre. This paper generalizes the aforementioned result for  $ZA$ -loops. It is proved that

for a CML  $L$  with multiplication group  $\mathfrak{M}$  the following statements are equivalent: 1) the CML  $L$  belongs to class  $\Omega$ ; 2) the centralizer of some finitely generated subloop of  $L$  belongs to class  $\Omega$ ; 3) the group  $\mathfrak{M}$  belongs to class  $\Omega$ ; 4) the centralizer of some finitely generated subgroup of group  $\mathfrak{M}$  belongs to class  $\Omega$ .

We denote that the equity of statements 3) and 4) is false for arbitrary group  $\mathfrak{M}$ . In [6] it is constructed an example of locally nilpotent group of infinite rank in which the centralizer is an infinite cyclic subgroup and has a finite rank. The centre of this group is different from unity.

Let  $G$  be a group. The commutator  $[a, b]$  of the elements  $a, b \in L$  is defined by equality  $[a, b] = a^{-1}b^{-1}ab = a^{-1}a^b$ . The identity

$$[xy, zt] = [x, t]^y[y, t][x, z]^{yt}[y, z]^t \quad (1)$$

holds in  $G$ .

Let  $M$  be a subset and  $H$  be a subgroup of group  $G$ . The subgroup  $C_H(M) = \{x \in H \mid [x, y] = 1 \ \forall y \in M\}$  is called *centralizer* of subset  $M$  into subgroup  $H$ . If  $M = H = G$ , then the normal subgroup  $C_G(G)$  is called *centre* of group  $G$ . We will denote it by  $C(G)$ .

Let us bring some notions and results on the theory of the *commutative Moufang loops* (abbreviated *CMLs*) from [5], which are characterized by the identity  $x^2 \cdot yz = xy \cdot xz$ .

The *multiplicative group*  $\mathfrak{M}(L)$  of a CML  $L$  is the group generated by all the *translations*  $L(x)$ , where  $L(x)y = xy$ . The *associator*  $(a, b, c)$  of the elements  $a, b, c$  in CML  $L$  is defined by the equality  $ab \cdot c = (a \cdot bc)(a, b, c)$ . The identities:

$$(xy, u, v) = (x, u, v)((x, u, v), x, y)(y, u, v)((y, u, v), y, x), \quad (2)$$

$$(x, y, z) = (y^{-1}, x, z) = (y, x, z)^{-1} = (y, z, x) \quad (3)$$

hold in CML  $L$ .

The *centre*  $Z(L)$  of a CML  $L$  is the normal subloop  $Z(L) = \{x \in L \mid (x, y, z) = 1 \ \forall y, z \in L\}$ . The *upper central series* of the CML  $L$  is the series

$$1 = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \dots \subseteq Z_\alpha \subseteq \dots$$

of the normal subloops of the CML  $L$ , satisfying the conditions: 1)  $Z_\alpha = \sum_{\beta < \alpha} Z_\beta$  for the limit ordinal and 2)  $Z_{\alpha+1}/Z_\alpha = Z(L/Z_\alpha)$  for any  $\alpha$ . If the CML possesses a central series, then this loop is called *ZA-loop*. If the upper central series of the *ZA-loop* has a finite length, then the loop is

called *centrally nilpotent*. The least of such a length is called the *class* of the central nilpotence.

By analogy it is defined the notion of *ZA-group* with the help of the notion of centre of group.

**Lemma 1 (Bruck-Slaby Theorem).** *Let  $n$  be a positive integer,  $n \geq 3$ . Then every commutative Moufang loop  $L$  which can be generated by  $n$  elements is centrally nilpotent of class at most  $n - 1$ .*

**Lemma 1<sup>0</sup>.** *The multiplication group of any CML is locally nilpotent.*

**Lemma 2.** *For any CML  $L$  with centre  $Z(L)$  the quotient loop  $L/Z(L)$  is locally finite 3-loop of exponent 3 and is finite if  $L$  is finitely generated.*

**Lemma 2<sup>0</sup>.** *Let  $L$  be a CML with multiplication group  $\mathfrak{M}$  and let  $C(\mathfrak{M})$  be the centre of  $\mathfrak{M}$ . Then the quotient group  $\mathfrak{M}/C(\mathfrak{M})$  is locally finite 3-group and is finite if  $L$  is finitely generated.*

The following concept is the natural generalization of the concept of centre. Let  $M$  be a subset and  $H$  be a subloop of CML  $L$ . The set  $Z_H(M) = \{x \in H \mid x \cdot yz = xy \cdot z \quad \forall y, z \in M\}$  is called *centralizer* of subset  $M$  into subloop  $H$ .  $Z_H(M)$  is a subloop of  $L$  [7].

**Lemma 3.** *A centrally nilpotent CML  $L$  belongs to class  $\Omega$  if and only if the centralizer of some of its finitely generated subloop  $H$  also belongs to class  $\Omega$ .*

**Proof.** The necessity of lemma is obvious. To prove the sufficiency, it is enough to achieve by induction by class of central nilpotence of CML  $L$ . We will prove this only for the case of rank finiteness, as the proof of other cases of class  $\Omega$  follows the same pattern.

Let the subloop  $H$  be generated by elements  $a_1, a_2, \dots, a_n$  and let the centralizer  $Z_L(H)$  have a finite rank. We will suppose that the CML  $L$  is non-associative, as for abelian groups (centrally nilpotent CML of class  $k = 1$ ) the statement holds.

Let  $Z$  be the centre of CML  $L$  and let  $k$  be the class of central nilpotence. Obviously,  $Z \subseteq Z_L(H)$ , hence the rank of  $Z$  is finite. As the quotient loop  $L/Z$  is centrally nilpotent of class  $k - 1$ , then by inductive assertion the rank of  $L/Z$  will be finite if the centralizer  $D/Z$  of image  $HZ/Z$  of subloop  $H$  into quotient loop  $L/Z$  has a finite rank. We will prove this.

Let  $x, y \in D$ , and let  $A_i = \{a_{i_1}, a_{i_2}\}, 1 = 1, 2, \dots, t$ , be an arbitrary fixed pair of elements  $a_{i_1}, a_{i_2} \in A$ . We have  $(x, a_{i_1}, a_{i_2}) \in Z$ , then by (1)  $(xy, a_{i_1}, a_{i_2}) = (x, a_{i_1}, a_{i_2})(y, a_{i_1}, a_{i_2})$ . This equality shows that the mapping  $x \rightarrow (x, a_{i_1}, a_{i_2})$  is a homomorphism of  $D$  into  $Z$ . For each  $A_i$  we consider the homomorphisms  $\varphi_i(x) = (x, a_{i_1}, a_{i_2}), x \in D$ . Obviously,  $\ker \varphi_i = Z_D(A_i)$ . Then the quotient loop is an abelian group of finite rank. The direct product

$\prod_{i=1}^t D/Z_D(A_i)$  also has a finite rank. By (1), (2) it is easy to see that  $\bigcap_{i=1}^t Z_D(A_i) = Z_D(A)$ .

Analogously to Remak Theorem for groups [8] it may be proved that the quotient loop  $D/Z_D(H)$  is isomorphic to subloop of direct product  $\prod_{i=1}^t D/Z_D(A_i)$ . Then  $D/Z_D(H)$  has a finite rank. As  $Z_D(H) \subseteq Z_L(H)$  then  $Z_D(H)$  also has a finite rank. Hence the CML  $D$  also has a finite rank.

Consequently,  $L/Z$  is a CML of finite rank. We mentioned above that the centre  $Z$  has a finite rank. Then CML  $L$  also has a finite rank. This completes the proof of Lemma 3.

**Lemma 3<sup>0</sup>** [6]. *A nilpotent group belongs to class  $\Omega$  if and only if the centralizer of some its finitely generated subloop belongs to class  $\Omega$ .*

**Lemma 4.** *Let  $H$  be a finitely generated subloop of CML  $L$ . If the centralizer  $Z_L(H)$  belongs to class  $\Omega$  then the centralizer  $Z_{L/Z(L)}(Z(L)H/Z(L))$  belongs to class  $\Omega$ .*

**Proof.** In [9, 10] it is proved that for CML the condition of finitely generation and maximum condition for subloops are equivalent, and in [3] is proved that these conditions are equivalent with the maximum condition for associative subloops. Further, in [2] it is proved that for a CML the minimum condition for subloops and the minimum condition for associative subloops are equivalents, and for  $p$ -loops this conditions are equivalents with condition of finiteness rank [4].

Now, let  $Z(L)$  be the centre of CML  $L$ , let  $\bar{L} = L/Z(L)$ ,  $\bar{H} = Z(L)H/Z(L)$ , and we suppose that the centralizer  $Z_{\bar{L}}(\bar{H})$  does not belong to class  $S$ . By Lemma 2 the quotient loop  $L/Z(L)$  satisfies the identity  $x^3 = 1$ . Then by the aforementioned,  $Z_{\bar{L}}(\bar{H})$  contains an infinite elementary abelian 3-group  $\bar{B}$ , which decomposes into a direct product of cyclic groups of order 3. Let  $A/Z(L) = \bar{A} = \bar{A}_1 \times \bar{A}_2 \times \dots \times \bar{A}_i \times \dots$  be the maximal subgroup of  $\bar{B}$  regarding to property  $\bar{A}_i \not\subseteq \bar{H}$ , and let  $\bar{A}_i = \langle \bar{a}_i \rangle$ . We denote by  $M(\bar{a}_i)$  a maximal subloop of CML  $\bar{R} = \langle \bar{A}, \bar{H} \rangle$  such that  $\bar{a}_i \notin M(\bar{a}_i)$ . As the element  $\bar{a}_i$  has order 3 then  $\bar{A}_i \cap M(\bar{a}_i) = \bar{1}$ . Every maximal subloop of CML is normal in this CML [10]. Let  $I(\bar{R})$  be the inner mapping group of CML  $\bar{R}$ . Every inner mapping of CML is an automorphism of this CML [5]. Then  $I(\bar{R})\bar{A}_i = \bar{A}_i$ . Hence the subloop  $\bar{A}_i$  is normal in  $\bar{R}$ . In [12] it is proved that if an element of order 3 of CML generates a normal subloop then this element belongs to the centre of this CML. Hence  $\bar{A}_i \subseteq Z(\bar{R})$ .  $\bar{A} \subseteq Z(\bar{R})$ . From here it follows that  $\bar{R} = \bar{A}\bar{H}$ . But  $\bar{A} \cap \bar{H} = \bar{1}$ . Then from  $I(\bar{R})\bar{A} = \bar{A}$  it follows that  $I(\bar{R})\bar{H} = \bar{H}$ , i.e. the subloop  $\bar{H}$  is normal in  $\bar{R}$ . Consequently,  $\bar{R} = \bar{A} \times \bar{H}$ .

The subloop  $H$  is finitely generated. Then by Lemma 1 its is centrally

nilpotent. The subloop  $\overline{H}$  is also centrally nilpotent. Then and subloop  $\overline{R}$  is centrally nilpotent. As  $\overline{B} \subseteq \overline{R}$  and  $\overline{B}$  do not belong to class  $S$  then  $\overline{R}$  also does not belong to class  $S$ . The inverse image of  $\overline{R}$  under the homomorphism  $L \rightarrow L/Z(L)$  is  $AH$ . This CML is centrally nilpotent and does not belong to class  $S$ . Then by Lemma 3 the centralizer  $Z_{AH}(H)$  does not belong to class  $S$ . We get a contradiction, as  $Z_{AH}(H) \subseteq Z_L(H)$  and  $Z_L(H)$  belong to class  $S$ . Consequently, the centralizer  $Z_{\overline{L}}(\overline{H})$  belongs to class  $S$ . This completes the proof of Lemma 4.

**Lemma 4<sup>0</sup>.** *Let  $\mathfrak{N}$  be a finitely generated subgroup of multiplication group  $\mathfrak{M}$  of CML  $L$ . If the centralizer  $C_{\mathfrak{M}}(\mathfrak{N})$  belongs to class  $\Omega$  then the centralizer  $C_{\mathfrak{M}/C(\mathfrak{M})}(C(\mathfrak{M})\mathfrak{N}/C(\mathfrak{M}))$  also belongs to class  $\Omega$ .*

**Proof.** In [3] it is proved that the finiteness of generators, the maximum condition for subgroups and maximum condition for abelian subgroups are equivalent for group  $\mathfrak{M}$ . Further, in [2] it is proved that for  $\mathfrak{M}$  the minimum condition for subgroups and the minimum condition for abelian subgroups are equivalent, and for  $p$ -groups these conditions are equivalent with the condition of finiteness rank [4, 11].

We suppose that the centralizer  $\mathfrak{B}/C(\mathfrak{M}) = C_{\mathfrak{M}/C(\mathfrak{M})}(C(\mathfrak{M})\mathfrak{N}/C(\mathfrak{M}))$  does not belong to class  $\Omega$ . Then by the first paragraph  $\mathfrak{B}/C(\mathfrak{M})$  contains an abelian subgroup  $\mathfrak{A}/C(\mathfrak{M})$  which does not belongs to class  $\Omega$ . From definition of centralizer it is follows that  $[\mathfrak{a}, \mathfrak{n}] = 1$  for  $\mathfrak{a} \in \mathfrak{A}, \mathfrak{n} \in \mathfrak{N}$ . Using this, it is easy to see that the product  $\mathfrak{A}\mathfrak{N}$  is a group. Moreover,  $\mathfrak{A}$  is abelian,  $\mathfrak{N}$  is nilpotent, then also using (1) we prove that  $\mathfrak{A}\mathfrak{N}$  is a nilpotent group.  $\mathfrak{A}$  does not belongs to class  $\Omega$  then and  $\mathfrak{A}\mathfrak{N}$  does not belongs to class  $\Omega$ . Then by Lemma 3<sup>0</sup> the centralizer  $C_{\mathfrak{A}\mathfrak{N}}(\mathfrak{N})$  does not belongs to class  $\Omega$ .  $C_{\mathfrak{M}}(\mathfrak{N})$  also does not belongs to class  $\Omega$  as  $C_{\mathfrak{A}\mathfrak{N}}(\mathfrak{N}) \subseteq C_{\mathfrak{M}}(\mathfrak{N})$ . We get a contradiction. Consequently,  $C_{\mathfrak{M}/C(\mathfrak{M})}(C(\mathfrak{M})\mathfrak{N}/C(\mathfrak{M}))$  belongs to class  $\Omega$ , as required.

**Lemma 5.** *Let  $H$  be a finitely generated subloop of CML  $L$ . If the centralizer  $Z_L(H)$  belongs to class  $\Omega$  then the centre  $Z(L)$  of CML  $L$  is different from unity.*

**Proof.** If  $a \in L$  is an element of infinite order then by Lemma 2  $1 \neq a^3 \in Z(L)$ . Then we will suppose that  $L$  is a periodic CML. In this cases  $L$  decomposes into a direct product of its maximal  $p$ -subloops  $L_p$ , in addition  $L_p$  belongs to the centre  $Z(L)$  under  $p \neq 3$ . Hence to prove Lemma 3 it is sufficient to suppose that  $L$  is a 3-loop. By Lemma 1 every finitely generated CML is centrally nilpotent, then we will suppose that CML  $L$  is infinite.

We remind that a system  $\{G_\alpha\}$  ( $\alpha \in I$ ) of subloops of loop  $G$  is a *local system* if the union  $\bigcup_{\alpha \in I} G_\alpha$  coincides with  $G$  and every two members of this

system are contained in a certain third member of this system. Using the definition of local system it is easy to prove the statement: if  $\{G_\alpha\}$  ( $\alpha \in I$ ) is some local system of loop  $G$  and  $I = I_1 \cup I_2 \cup \dots \cup I_k$  is a certain partition of the set of indices  $I$  into a finite number of subsets,  $I_j$ ,  $j = 1, 2, \dots, k$ , then at least on subset  $I_j$  corresponds to the set of subloops  $\{G_\beta\}$ ,  $\beta \in I_j$ , which will also be a local system for loop  $G$ .

Let now  $\{L_\alpha\}$ ,  $\alpha \in I$ , be the local system of all finitely generated subloops of CML  $L$ , which contain the subloop  $H$ . By Lemma 1  $Z(L_\alpha) \neq \{1\}$ . For each  $\alpha \in I$  we fixed an arbitrary non-unitary element  $a_\alpha$  from centre  $Z(L_\alpha)$  and let  $K$  be the subloop of  $L$  generated by all  $a_\alpha$ ,  $\alpha \in I$ . From  $K \subseteq Z_L(H)$  it follows that  $K$  belongs to class  $\Omega$ . We suppose that 3-subloop  $K$  has a finite rank. Then by [4],  $K$  satisfies the minimum condition for its subloops and by [2]  $K = R \times T$ , where  $R \subseteq Z(L)$  and  $T$  is a finite subloop. Further, any periodic CML is locally finite [5]. Hence to prove Lemma 3 it is sufficient to consider that  $K$  is a finite subloop.

Further, let us decompose the set of indices  $I$  into a finite number of subsets  $I = I_1 \cup I_2 \cup \dots \cup I_k$  by rule:  $\beta, \gamma \in I_j$  if and only if  $a_\beta = a_\gamma$ . According to the aforementioned statement we have received that at least one of the subsets  $I_j$  (e.g.  $I_1$ ) corresponds to the subset of subloops  $L_\alpha$ ,  $\alpha \in I_1$ , which will be a local system for CML  $L$ . Next, let us fix index  $\alpha \in I_1$ , and consider the set of indices  $S \subseteq I_1$ , such that  $L_\alpha \subseteq L_\beta$ ,  $\beta \in S$ . We notice that the set  $S$  corresponds to the set of subloops  $\{L_\beta\}$ ,  $\beta \in S$ , which will be also a local system for CML  $L$ . Let us denote the value of corresponding members by  $b$ ,  $b = a_\alpha = a_\beta = \dots$ . Then  $b \in Z(L_\beta)$  for all  $\beta \in S$  and, consequently,  $b \in Z(L)$ . This completes the proof of Lemma 5.

**Lemma 5<sup>0</sup>.** *Let  $\mathfrak{N}$  be a finitely generated subgroup of multiplication group  $\mathfrak{M}$  of CML  $L$ . If the centralizer  $C_{\mathfrak{M}}(\mathfrak{N})$  belongs to class  $\Omega$  then the centre  $C(\mathfrak{M})$  of group  $\mathfrak{M}$  is different from unity.*

**Proof.** We will prove Lemma 5<sup>0</sup>, using the same pattern as for Lemma 5. If  $a \in \mathfrak{M}$  is an element of infinite order then by Lemma 2<sup>0</sup>  $a^{3^k} \in C(\mathfrak{M})$  for some integer  $k$ . Then we will suppose that  $\mathfrak{M}$  is a periodic group. In this case  $\mathfrak{M}$  decomposes into a direct product of its maximal  $p$ -subgroups  $\mathfrak{M}_p$ , in addition  $\mathfrak{M}_p$  belongs to the centre  $C(\mathfrak{M})$  under  $p \neq 3$ . Hence to prove Lemma 5<sup>0</sup> it is sufficient to suppose that  $\mathfrak{M}$  is a 3-group. By Lemma 1<sup>0</sup> a finitely generated subgroup of multiplication group of CML is nilpotent, then we will suppose that  $\mathfrak{M}$  is infinite.

Let  $\{\mathfrak{M}_\alpha\}$ ,  $\alpha \in I$ , be a local system of all finitely generated subgroups of group  $\mathfrak{M}$ , which contain the subgroup  $\mathfrak{N}$ . By Lemma 1<sup>0</sup>  $C(\mathfrak{M}_\alpha) \neq \{1\}$ . For each  $\alpha \in I$  we fixed an arbitrary non-unitary element  $a_\alpha$  from centre

$C(\mathfrak{M}_\alpha)$  and let  $\mathfrak{K}$  be the subgroup of  $\mathfrak{M}$  generated by all  $a_\alpha$ ,  $\alpha \in I$ . From  $\mathfrak{K} \subseteq C_{\mathfrak{M}}(\mathfrak{M})$  it follows that  $\mathfrak{K}$  belong to class  $\Omega$ . We suppose that 3-subgroup  $\mathfrak{K}$  has a finite rank. Then by [4, 11]  $\mathfrak{K}$  satisfies the minimum condition for its subgroups and by [2]  $\mathfrak{K} = \mathfrak{R} \times \mathfrak{T}$ , where  $\mathfrak{R} \subseteq C(\mathfrak{M})$  and  $\mathfrak{T}$  is a finite subloop. Any periodic multiplication group of CML is locally finite [5]. Then to prove Lemma 5<sup>0</sup> it is sufficient to consider that  $\mathfrak{K}$  is a finite subgroup.

Further, let us decompose the set of indices  $I$  into a finite number of subsets  $I = I_1 \cup I_2 \cup \dots \cup I_k$  by the rule:  $\beta, \gamma \in I_j$  if and only if  $a_\beta = a_\gamma$ . According to the aforementioned statement we have received that at least one of the subsets  $I_j$  (e.g.  $I_1$ ) corresponds to the subset of subgroups  $\mathfrak{M}_\alpha$ ,  $\alpha \in I_1$ , which will be a local system for group  $\mathfrak{M}$ . Next, let us fix index  $\alpha \in I_1$ , and consider the set of indices  $S \subseteq I_1$ , such that  $\mathfrak{M}_\alpha \subseteq \mathfrak{M}_\beta$ ,  $\beta \in S$ . We notice that the set  $S$  corresponds to the set of subgroups  $\{\mathfrak{M}_\beta\}$ ,  $\beta \in S$ , which will be also a local system for group  $\mathfrak{M}$ . Let us denote the value of corresponding members by  $b$ ,  $b = a_\alpha = a_\beta = \dots$ . Then  $b \in C(\mathfrak{M}_\beta)$  for all  $\beta \in S$  and, consequently,  $b \in C(\mathfrak{M})$ . This completes the proof of Lemma 5<sup>0</sup>.

**Theorem.** *For a CML  $L$  with multiplication group  $\mathfrak{M}$  the following statements are equivalent:*

- 1) *the CML  $L$  belongs to class  $\Omega$ ;*
- 2) *the centralizer of some finitely generated subloop  $H$  of  $L$  belongs to class  $\Omega$ ;*
- 3) *the group  $\mathfrak{M}$  belongs to class  $\Omega$ ;*
- 4) *the centralizer of some finitely generated subgroup  $\mathfrak{N}$  of group  $\mathfrak{M}$  belongs to class  $\Omega$ .*

**Proof.** 2)  $\Rightarrow$  1). Let the centralizer  $Z_L(H)$  belong to class  $\Omega$ . We denote  $\overline{L} = L/Z(L)$ ,  $\overline{H} = HZ(L)/Z(L)$ . Any periodic CML is locally finite [5]. Then from Lemma 2 it follows that the subloop  $\overline{H}$  is finite. From Lemmas 4, 5 it follows that the upper central series of CML  $\overline{L}$  has a form  $\overline{1} \subset Z_1(\overline{L}) \subset \dots \subset Z_k(\overline{L}) \subset \dots$ , and for a natural number  $n$   $Z_n(\overline{L}) \neq Z_{n+1}(\overline{L})$  if  $Z_n(\overline{L}) \neq \overline{L}$ . As  $\overline{H}$  is finite, then for some  $k$   $\overline{H} \not\subseteq Z_{k-1}(\overline{L})$ , but  $\overline{H} \subseteq Z_k(\overline{L})$ . Then  $Z_{\overline{L}}(\tilde{H}) = \tilde{L}$ , where  $\tilde{L} = \overline{L}/\overline{Z_k}$ ,  $\tilde{H} = \overline{H}\overline{Z_k}/\overline{Z_k}$ . By Lemma 2  $\tilde{L}$  is a 3-loop and by Lemma 4  $\tilde{L}$  belongs to class  $\Omega$ . In [4] it is proved that the minimum condition for subloops and the condition of finiteness rank are equivalent for CML  $\tilde{L}$ . In this case  $\tilde{L} = \tilde{R} \times \tilde{T}$ , where  $\tilde{R} \subseteq Z(\tilde{L})$ , and  $\tilde{T}$  is a finite CML which by Lemma 1 is centrally nilpotent. Then CML  $\tilde{L} = \overline{L}/\overline{Z_k} = (L/Z)/(Z_k/Z) \cong L/Z_k$  is also centrally nilpotent. From central nilpotence of  $L/Z_k$  it follows the central nilpotence of  $L$ . By Lemma 3  $L$  belongs to class  $\Omega$ . Consequently, the implication 2)  $\Rightarrow$  1) hold.

4)  $\Rightarrow$  3). Let the centralizer  $C_{\mathfrak{M}}(\mathfrak{N})$  belong to class  $\Omega$ . We denote  $\overline{\mathfrak{M}} = \mathfrak{M}/C(\mathfrak{M})$ ,  $\overline{\mathfrak{N}} = \mathfrak{N}C(\mathfrak{M})/C(\mathfrak{M})$ . Any periodic multiplication group of CML is locally finite [5]. Then from Lemma 2<sup>0</sup> it follows that the subgroup  $\overline{\mathfrak{N}}$  is finite. From Lemmas 4<sup>0</sup>, 5<sup>0</sup> it follows that the upper central series of group  $\overline{\mathfrak{M}}$  has a form  $\overline{1} \subset C_1(\overline{\mathfrak{M}}) \subset \dots \subset C_k(\overline{\mathfrak{M}}) \subset \dots$ , and for a natural number  $n$   $C_n(\overline{\mathfrak{M}}) \neq C_{n+1}(\overline{\mathfrak{M}})$  if  $C_n(\overline{\mathfrak{M}}) \neq \overline{\mathfrak{M}}$ . As  $\overline{\mathfrak{N}}$  is finite, then for some  $k$   $\overline{\mathfrak{N}} \not\subseteq C_{k-1}(\overline{\mathfrak{M}})$ , but  $\overline{\mathfrak{N}} \subseteq C_k(\overline{\mathfrak{M}})$ .

Then  $C_{\tilde{\mathfrak{M}}}(\tilde{\mathfrak{N}}) = \tilde{\mathfrak{M}}$ , where  $\tilde{\mathfrak{M}} = \overline{\mathfrak{M}}/\overline{C_k}$ ,  $\tilde{\mathfrak{N}} = \overline{\mathfrak{N}C_k}/\overline{C_k}$ . By Lemma 2<sup>0</sup>  $\tilde{\mathfrak{M}}$  is a 3-loop and by Lemma 4<sup>0</sup>  $\tilde{\mathfrak{M}}$  belongs to class  $\Omega$ . In [4] it is proved that the minimum condition for subgroups and the condition of finiteness rank are equivalent for group  $\tilde{\mathfrak{M}}$ . In this case  $\tilde{\mathfrak{M}} = \tilde{\mathfrak{R}} \times \tilde{\mathfrak{Z}}$ , where  $\tilde{\mathfrak{R}} \subseteq C(\tilde{\mathfrak{M}})$ , and  $\tilde{\mathfrak{Z}}$  is a finite group which by Lemma 1<sup>0</sup> is nilpotent. Then group  $\tilde{\mathfrak{M}} = \overline{\mathfrak{M}}/\overline{C_k} = (\mathfrak{M}/C)/(C_k/C) \cong \mathfrak{M}/C_k$  is also nilpotent. From nilpotence of  $\mathfrak{M}/C_k$  it follows the nilpotence of  $\mathfrak{M}$ . By Lemma 3<sup>0</sup>  $\mathfrak{M}$  belongs to class  $\Omega$ . Consequently, the implication 4)  $\Rightarrow$  3) hold.

Hence  $C_{k+1}(\mathfrak{M}) = \mathfrak{M}$ . From here it follows that the group  $\overline{\mathfrak{M}}$  is nilpotent. Then and the group  $\mathfrak{M}$  is nilpotent and by Lemma 3<sup>0</sup>  $L$  belongs to class  $\Omega$ .

The implications 1)  $\Rightarrow$  2), 3)  $\Rightarrow$  4) are obvious.

The statements:  $L$  is finitely generated and the maximum conditions holds in  $L$  are equivalent for any CML  $L$  [9, 10]. Then the implications 1)  $\Leftrightarrow$  3) are proved in [2 – 4]. This completes the proof of the Theorem.

## References

- [1] P. Hall, *On the finiteness of certain soluble groups*. Proc. London Math. Soc., 1959, 9, 36, 595 – 622.
- [2] N. I. Sandu, *Commutative Moufang loops with minimum condition for subloops I*. Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2003, 3(43), 25 – 40.
- [3] A. Babiş, N. Sandu, *The commutative Moufang loops with maximum conditions for subloops*. Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2006, 2(51), 53 – 61.
- [4] A. Babiş, N. Sandu, *About commutative Moufang loops of finite special rank*. Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2006, 1(50), 92 – 100.



- [5] Bruck R. H., *A survey of binary systems*. Springer Verlag, Berlin-Heidelberg, 1958.
- [6] D. I, Zaicev, V. A. Onishchuk, *On locally nilpotent groups with centralizer satisfying finiteness condition*. Ukr. mat. журн., 1991, 43, 7 – 8, 1084 – 1087 (In Russian).
- [7] Sandu N. I.: *Commutative Moufang loops with finite classes of conjugate elements*. Mat. zametki, 2003, 73, 2, 269 – 280 (In Russian).
- [8] Kurosh A. G., *Group Theory*. Moscow, Nauka, 1967 (In Russian).
- [9] Evans T., *Identities and Relations in Commutative Moufang Loops*. J. Algebra, 31, 1974, 508 – 513.
- [10] Sandu N. I., *About Centrally Nilpotent Commutative Moufang Loops*. Quasigroups and Loops (Matem. issled., 51), Kishinev, 1979, 145 – 155 (In Russian).
- [11] Meagkova N. N. *About Groups of Finite Rank*. Izv. AN SSSR. Ser. matem., 1949, 13, 6, p. 495–512 (In Russian).
- [12] Sandu N. I., *Commutative Moufang Loops with Minimum Condition for Subloops II*. Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2004, N 2(45), p. 33–48.

Tiraspol State University (Moldova)

E-mail: aliona2010@yahoo.md; sandumn@yahoo.com